Sound, Trigonometry, and Fourier Mathematics

Mike Cirillo

Abstract

The aim of this paper is to provide an accessible and intuitive introduction to the basic physics and mathematics of sound waves. It requires no prior mathematics or physics knowledge beyond basic arithmetic, geometry, and high school physics. We begin with a description of simple harmonic motion, compound tones, a brief survey of the history of trigonometry, and then delve into the techniques of basic trigonometry and Fourier mathematics.

The mathematics portion of this paper starts with, more or less, first principles, then proceeds in a casually deductive manner to its conclusion. This is not a rigorous mathematical demonstration, but a friendly exploration and exposition. It is the hope of the author that this paper will provide an enjoyable and accessible introduction to those not ordinarily inclined toward science and mathematics.

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Enough exposition: let us begin with Simple Harmonic Motion.

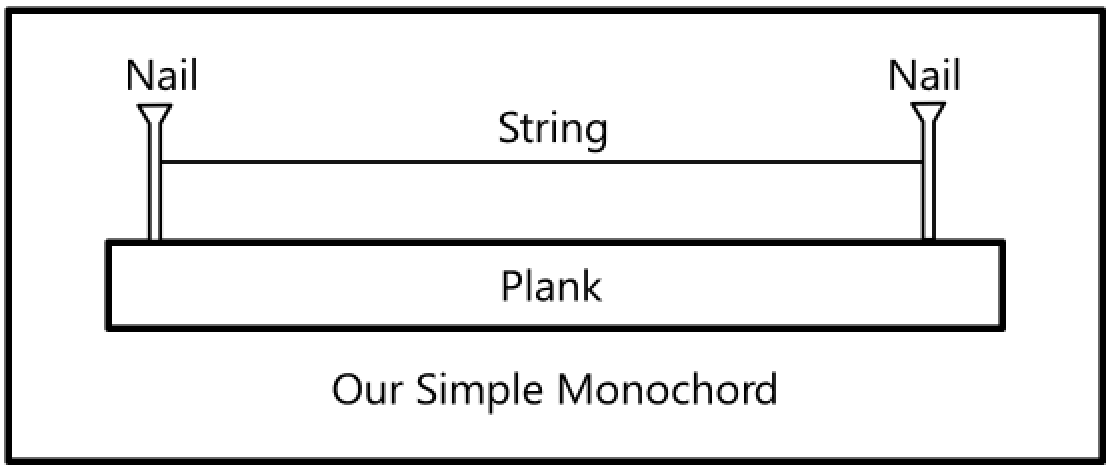
Albert Einstein enjoyed thought experiments. He would relax in a stuffed chair, smoke his pipe, and contemplate time and space. Sometimes he played the violin while meditating on the harmony of the universe. [1]

Let us follow his example. Feel free to relax in a comfortable chair. Pour yourself a cup of coffee if you would like. We begin by visualizing an experimental instrument of the ancient

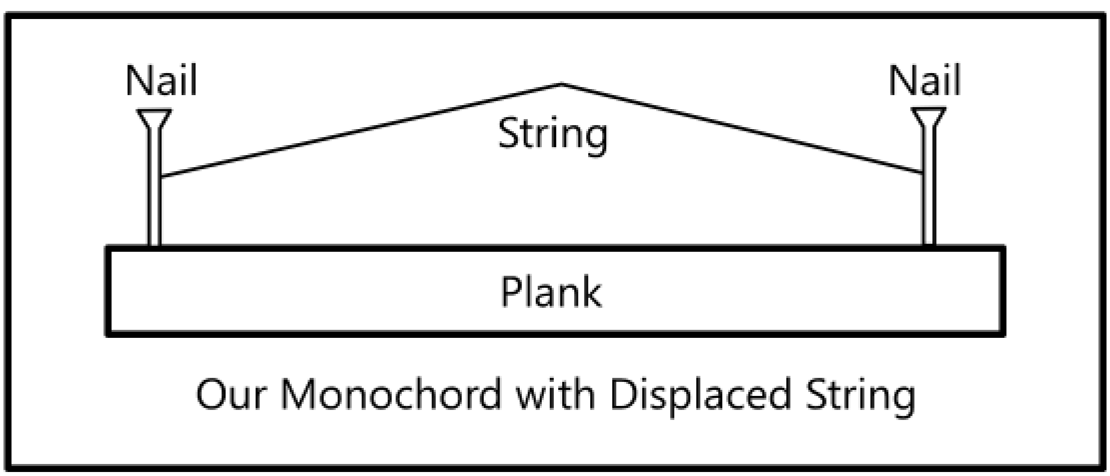
[1] Walter Isaacson, *Einstein: His Life and Universe* (New York, NY: Simon & Schuster, 2007), p. 14 and 438.

Greeks: the monochord. In choosing this instrument, we are honoring an intellectual inquiry with provenance to Pythagoras, the monochords inventor.[2]

Our monochord will be simpler than that of ancient Greece. Ours will be nothing more than a wooden plank with a violin string pulled tight between two nails. The illustration below should clarify.



The action begins: grasp the middle of the string, and pull upward.



[2] Cecil Adkins, "Monochord" in *Grove Music Online*. *Oxford Music Online*, http://www.oxfordmusiconline.com/subscriber/article/grove/music/18973 (accessed March 21, 2011).

What happens to the string? For this we detour into the work of the English scientist Robert Hooke (1635-1703). Hooke discovered that as a force causes an object to deform, and the deformation is within the limit of the objects elasticity, then the deformation of the object is proportional to the applied force.[3]

We will also be assisted by a contemporary of Hooke, Isaac Newton (1642 – 1726).[4]

Today, Isaac Newton is considered one of the greatest scientific minds in history. But in the year 1665, he was nobody. He had just finished his undergraduate work at Cambridge. The plague was tearing through London. As a precaution, the university closed and Newton went into seclusion at his family home in Woolthorpe. During this time Newton invented the calculus, created a mathematical theory of gravitation, demonstrated the binomial theorem, and wrote a major work on the theory of color.[5] Later in life, reflecting on his accomplishments, Newton wrote, "I do not know what I may appear to the world; but to myself I seem to have been only like a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all but undiscovered before me."[6]

We will borrow one of Newton's pretty shells: The Third Law of Motion. Wolfram|Alpha defines Newton's third law as, "Whenever a particle *A* exerts a force on another particle *B*, *B* simultaneously exerts a force on *A* with the same magnitude in the

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[3] Isaac Asimov, *Understanding Physics: Motion, Sound, and Heat* (New York, New York: NAL Penguin Inc., 1966), p. 49-50.

[4] Asimov, p. 23.

[5] Carl B. Boyer, A *History of Mathematics* (New York: John Wiley & Sons, Inc, 1968), p. 430-431.

[6] William Dunham, *Journey through Genius: The Great Theorems of Mathematics* (New York: Penguin Group, 1990), p. 183.

opposite direction. Furthermore, these two forces act along the same line."[7] Put simply: If you push on something, it will push back with equal force.

How does this law relate to our visualization? As we pull on the string, it pulls back. Therefore, as we apply more force and pull the string farther from equilibrium, it pulls back with equal and opposite force.

To make this more intuitive we will briefly anthropomorphize the system: The more we stretch the string, the more it wants to return to its original length. Therefore, it pulls back harder. The further we stretch the string, the harder it pulls back.

Relating this to our monochord: as we pull the string farther from equilibrium, the restoring force increases. Of course, if we pull hard enough, the string will eventually break. At that point, Hooke's law would no longer apply. In our visualization, we will be careful not to break the string.

Release the string. The restorative force begins accelerating the string back toward equilibrium. As the string gets closer and closer to equilibrium, the force pulling it back lessens. Therefore, although the speed increases all the way back to equilibrium, it increases by ever decreasing amounts. When the string reaches equilibrium, the restorative force will be zero.[8] For a brief moment, the string stops accelerating.

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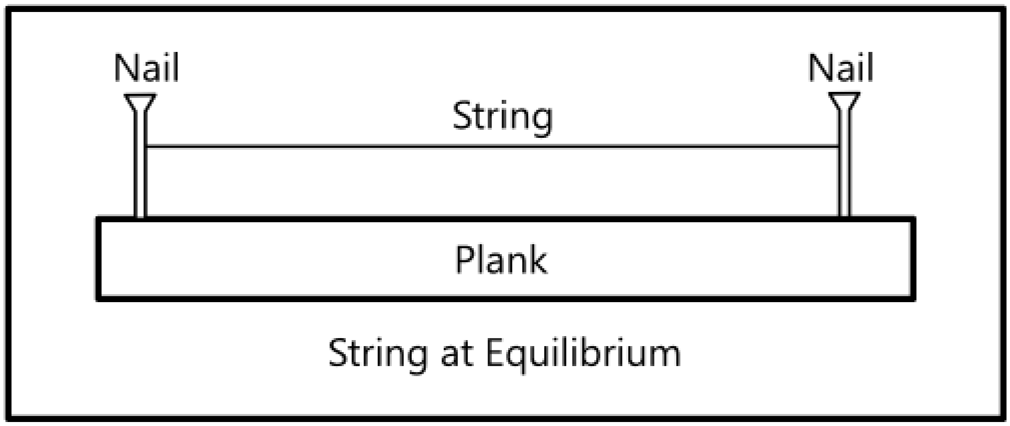
[7] Wolfram|Alpha (April 26, 2011) http://www.wolframalpha.com/input/?i=newton+third+law+of+motion
[8] Asimov, p. 101-103.



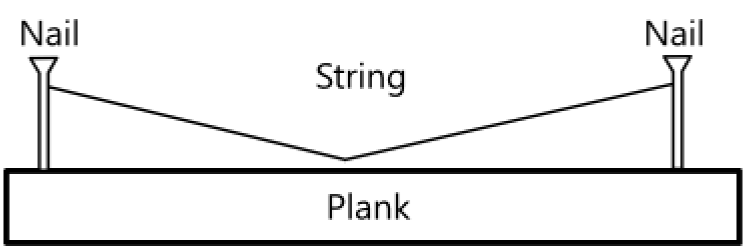
Does the string then simply stop moving at equilibrium? To answer this question we will look at a fundamental law of physics called the Conservation of Energy. Wolfram|Alpha defines Conservation of Energy as, "the fundamental principle of physics that the total energy of an isolated system is constant despite internal changes."[9] It was an English brewer, James Prescott Joule (1818-1889) who, as a hobby, thoroughly tested this premise. In honor of his efforts, the *International System of Units* named the unit for work and energy after him: the Joule. The German physicist, biologist, and music theorist, Hermann von Helmholtz (1821-1894), formally stated the law and won its acceptance by his scientific peers in 1847.[10]

Energy never disappears; it just manifests itself in different ways. Although energy can take a multitude of forms, for our purposes we will consider only two: Elastic Potential Energy and Kinetic Energy. Wolfram|Alpha defines Elastic Potential Energy as "potential energy that is stored when a body is deformed (as in a coiled spring)."[11] Kinetic Energy is defines as "the mechanical energy that an object has by virtue of its motion."[12] Therefore, when the string is

[9] Wolfram|Alpha (April 26,2011) http://www.wolframalpha.com/input/?i=conservation+of+energy
[10] Asimov, p. 99.
[11] Wolfram|Alpha (April 26, 2011) http://www.wolframalpha.com/input/?i=elastic+potential+energy
[12] Wolfram|Alpha (April 26, 2011) http://www.wolframalpha.com/input/?i=kinetic+energy+definition

pulled but not yet released, it possesses all elastic potential energy and no kinetic energy. When the string is released, and the string accelerates and the deformation shrinks, the potential energy gradually converts into kinetic energy. This conversion continues until equilibrium. At equilibrium, all the elastic potential energy has been completely changed into kinetic energy.

Does the string stop? What happens to all this kinetic energy? It cannot just disappear. This Kinetic energy moves the string past equilibrium. As soon as the string is no longer at equilibrium, Hooke's law kicks in, exerting a restorative force opposite to the direction of motion, proportional to the amount of displacement. The further the string moves past equilibrium, the stronger becomes the restorative force. This restorative force gradually decelerates the string, converting its kinetic energy back into potential energy. The string keeps deforming, gradually slowing down, until it reaches a deformation equal and opposite to the original deformation. At that moment, all the energy has been converted from kinetic to potential.



Monochord with String Deformed in Opposite Direction

Hooke's Law, once again, begins accelerating the string back toward equilibrium. If we disregard gravity and friction, the string will cycle back and forth, from opposite deformation to opposite deformation, indefinitely. Of course, in the real world, friction gradually converts the kinetic energy into heat energy. The heat energy dissipates into the environment and the motion slows and stops.

This type of motion is called periodic motion. Isaac Asimov explains periodic motion: "Whenever a motion goes through a series of repetitive submotions, each with a period of its own, the motion is said to be a *periodic motion*, particularly when the individual periods are equal." Asimov describes the period of the motion as "the time it takes to move from the extreme point on one side to the extreme point on the other and back."[13]

The term "frequency" is used to describe how many periods per second. The correct unit for frequency, as decreed by the *Systéme Internationale* in 1960, is the Hertz, after the German physicist Heinrich R. Hertz (1857-94).[14] Hertz is defined: "Something which makes 1 complete vibration every second has a frequency of 1 hertz (or 1 Hz)".[15] The frequency of the vibration in a sound wave determines its musical pitch. Hermann L. F. Helmholtz writes in, *On the Sensations of Tone as a Physiological Basis for the Theory of Music*, "*Pitch depends solely on the length of time in which each single vibration is executed*, *or*, which comes to the same thing, *on the number of vibrations completed in a given time*."[16] The frequency most musicians are familiar with is 440 Hz, the pitch A_4.[17]

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[13] Asimov, p. 103-104.
[14] John Borwick, "Hertz" in *The Oxford Companion to Music, Oxford Music Online*. http://www.oxfordmusiconline.com/subscriber/article/opr/t114/e3231 (accessed March 30, 2011)
[15] Ian Johnston, *Measured Tones* (New York, NY: IOP Publishing Ltd., 1980), p. 35.
[16] Hermann L. F. Helmholtz, *On the Sensation of Tone* (New York: Dover Publications, 1954), p. 11.
[17] Johnston, p. 39.

Now that we have finished our thought experiment, we return to the real world. Most musical instruments are not limited to this kind of simple harmonic motion. Musical sounds amalgamate many simple waves. Helmholtz describes these more complicated waves as *compound tones*.[18] James Jeans explains the mechanical aspect: "There is a very general principle in mechanics, which asserts that when any structure whatever is set into vibration – provided only that the displacement of each particle is small – the motion of every particle is either a simple harmonic motion or else is a more complicated motion which results from superposing a number of simple harmonic motions, one for each vibration which is in progress."[19]

We can precisely describe and create compound tones using mathematics. To do this requires two fields: trigonometry and Fourier analysis. We will begin with trigonometry and follow with Fourier analysis.

The word "Trigonometry" is a mash-up of the Greek word for triangle "trigonon" and the Greek word for measure "metron".[20] Therefore, the intimidating name "trigonometry" is simply a Greek term for the study of measuring triangles. It first appeared in the title of a book, *Trigonometry, or, Concerning the Properties of Triangles, in five books* (1595), by the German clergyman and mathematician Bartholomäus Pitiscus (1561-1613).[21]

The general concept of measuring triangles existed as a simple proto-trigonometry long before Trigonometry proper. The Rhind Papyrus, a scroll written almost 4000 years ago in

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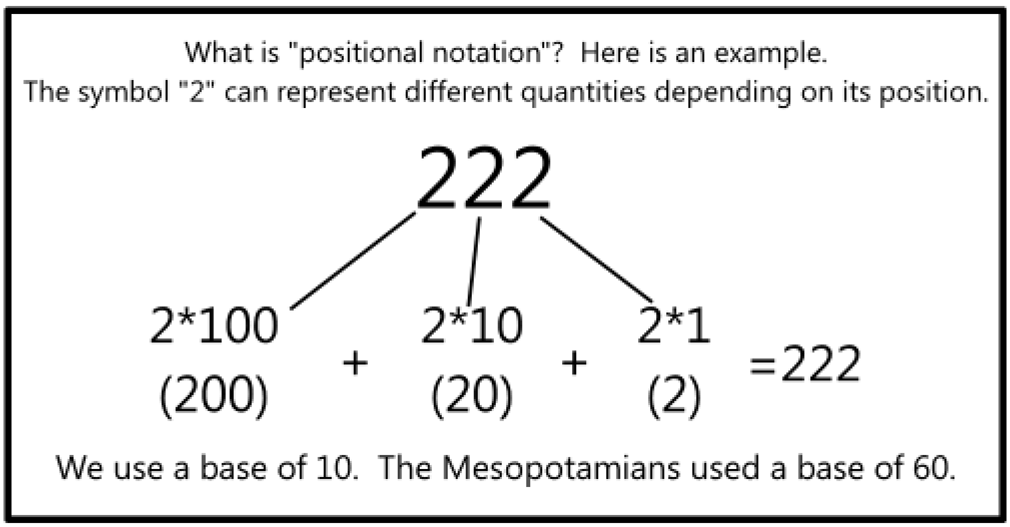
[18] Helmholtz, p. 23
[19] Sir James Jeans, *Science and Music* (New York: Dover Publications, 1968), p. 32-33.
[20] Eli Maor, *Trigonometric Delights* (New Jersey: Princeton University Press, 1998), p. 20.
[21] Maor, p. 40.

Egypt, describes a method of triangulating, from a desired base and height, the appropriate slopes for building a pyramid.[22]

Concurrent civilizations in Mesopotamia practiced the most advanced mathematics of the time. Mesopotamia means "land between the rivers" (Tigris and Euphrates) and is approximate to modern day Iraq.[23] They developed numeration based on positional notation, they invented the concept of a placeholder symbol, and they created computational methods based on positional manipulation. Relevant to our inquiry, the Mesopotamians also invented a rudimentary proto-trigonometry in which they explored the relationships between the sides of right triangles, and found numerous Pythagorean triples.[24]



During the years between 800 BC and 800 AD, the intellectual focus transitioned from Egypt and Mesopotamia to Greek civilizations emerging along the banks of the Mediterranean Sea. This time span is called the Thalassic Age ("sea" age). William Dunham describes the

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22 Maor, p. 3-9.

23 Marian H. Feldman "Mesopotamia" in *The Oxford Encyclopedia of Ancient Egypt*. http://www.oxford-ancientegypt.com/entry?entry=t176.e0453 (Accessed April 7, 2011).

24 Boyer, p. 26-46.

Greeks: "Engaged in a thriving commerce, both within their own lands and across the Mediterranean, the Greeks developed into a mobile, adventuresome people, relatively prosperous and sophisticated, and considerably more independent in thought and action than the western world had seen before. These curious, free-thinking merchants were much less likely to submit meekly to authority. Indeed, with the development of Greek democracy, the citizens became the authority…To such individuals, everything was open to debate and analysis, and ideas were not about to be accepted with a passive, unquestioned obedience."[25]

These early Greek mathematicians transformed the rudimentary mathematics of Egypt and Babylonia into a powerful deductive framework. As Plato said, "Whatever we Greeks receive, we improve and perfect."[26] Carl B. Boyer writes, "The Greeks were far from hesitant in taking over elements of foreign cultures, else they would never have learned so quickly how to advance beyond their predecessors; but everything they touched, they quickened."[27]

The most famous and influential of the early Greek mathematicians was Pythagoras (585 BC – 500 BC).[28] He is reputed to have studied in Egypt and Mesopotamia, perhaps even India.[29] Pythagoras founded a school in Southern Italy[30] and had a strict code of conduct for his students. Disciples were required to be celibate and could not "wear wool clothing, eat meats or beans except on the occasion of a religious sacrifice, touch a white cock, sit on a quart measure, walk on the high roads, use iron to stir a fire, or leave marks of ashes on a pot."

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[25] Dunham, p. 5.
[26] Morris Kline, *Mathematics in Western Culture* (New York: Oxford University Press, 1953), p. 24.
[27] Boyer, p. 50.
[28] Kline, *Mathematics in Western Culture*, p. 40.
[29] Boyer, p. 52
[30] Morris Kline, *Mathematical Thought from Ancient to Modern Times* (New York: Oxford University Press, 1972), p. 27.

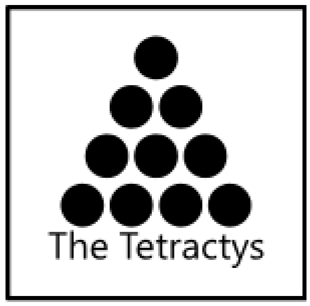
Members were also required to take a vow of secrecy.[31] In spite of the eccentricities and mysticism, Pythagoras is a monumental figure in math, science, music theory, and philosophy. His experiments in musical harmony were the earliest studies of acoustics and possibly the earliest quantitative physical laws.[32] The very words "philosophy" ("love of wisdom") and "mathematics" ("that which is learned") are said to have originated with Pythagoras. The motto of the Pythagorean School, "All is number," would not be out of place coming from a modern mathematical physicist. Pythagoras is often credited for starting mathematics down the road on its journey toward formal deduction. The philosopher Proclus (410 CE-485 CE) writes, "Pythagoras…transformed this science into a liberal form of education, examining its principles from the beginning and probing the theorems in an immaterial and intellectual manner."[33]

Pythagoras's famous namesake theorem, $a^2+b^2=c^2$, describes a relationship between the sides of rights triangles. As mentioned before, this theorem is of mysterious birth. There is evidence that it was familiar to the Babylonians sometime between 1800 BC and 1600 BC. This was over a thousand years before Pythagoras.[34] It is not known what, if anything, Pythagoras actually contributed to this theorem.[35]

In Pythagorean number worship, ten was sacred. It was called the *tetractys*. It was "the number of the universe, including the sum of all the possible geometric dimensions. A single point is the generator of dimensions, two points determine a line of dimension one, three

[31] Kline, *Mathematics in Western Culture*, p. 40-41.
[32] Boyer, p. 60.
[33] Boyer, p. 51-54.
[34] Maor, p. 30.
[35] Boyer, p. 54.

points (not on a line) determine a triangle with area of dimension two, and four point (not in a plane) determine a tetrahedron with volume of dimension three; the sum of the numbers representing all dimensions therefore is the revered number ten."[36] The *tetractys* was represented as a triangle made up of ten points.[37]



The next major Greek school was The Academy in Athens. Plato founded The Academy in roughly 387 BC. Posted above the door was a warning: "Let no one ignorant of geometry enter here."[38] Plato was strongly influenced by the Pythagorean School and emphasized the importance of pure math, not for practical application, but as a means of conceptualizing the ideal universe. Morris Kline writes of the Platonic conception of mathematics: "It purifies the mind by drawing it away from the contemplation of the sensible and perishable to the eternal. The path to salvation, then, to the understanding of Truth, Beauty, and Goodness, led through mathematics. This study was an initiation into the Mind of God."[39]

Plato reveled in triangles. He conceptualized the five regular polyhedra – the "Platonic solids" as they were later called – as being built of triangles. Therefore, the five elements represented by these polyhedra – fire, water, earth, air, and the universal element – consisted

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[36] Boyer, p. 57-58.
[37] Siglind Bruhn, *The Musical Order of the World* (Hillsdale, NY: Pendragon Press, 2005), p. 65.
[38] Boyer, p. 93.
[39] Kline, *Mathematics in Western Culture*, p. 32-33.

of patterned triangles. This was his version of a grand unified theory.[40] Of course, today we know that the Universe is far more complex than this.

Plato was teacher to Aristotle, and Aristotle was tutor to Alexander the Great. Alexander, finding time between marching and conquering, founded the city of Alexandria and decreed it the new capital of the ancient world. Alexander died in 323 BC and Aristotle a year later.[41] A general in the Greek army, Ptolemy I, took over the Egyptian portion of the empire and decided to build, in Alexandria, the greatest school in the world. It would be a home for the Muses. It was called The Museum.[42]

The Museum employed, among many great scholars, the most famous math teacher in history, who wrote the most famous math textbook in history: The *Elements* by Euclid.[43] It is thought that Euclid studied at Plato's Academy before moving to Alexandria sometime around 300 BC.[44] Although the book does not contain trigonometry proper, many of the theorems are about triangles and lead the way towards trigonometry.[45]

The first real trigonometric table was computed by the astronomer, Hipparchus of Nicaea (180 BC – 125 BC). He required a table of trigonometric ratios for his astronomy work,

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[40] Boyer, p. 93-97.
[41] Boyer, p. 108.
[42] Kline, *Mathematics in Western Culture*, p. 60-61.
[43] Boyer, p. 111.
[44] Kline, *Mathematical Thought from Ancient to Modern Times*, p. 56.
[45] Boyer, p. 176.

and being that none existed, computed one.[46] For this, he is considered the father of trigonometry.[47]

Expanding on Hipparchus was Claudius Ptolemaeus (85 – 165). Ptolemaeus is often known as Ptolemy, although he is unrelated to the Ptolemy dynasty that took over Egypt after Alexander the Great died. His book, *The Almagest* (The Greatest), is a compendium of mathematical astronomy reflecting the scientific and mathematical state of the art at the time. Like most astronomers, Ptolemy depended on trigonometry for his work. He also wrote on music and geography, although his geographic estimations considerably underestimated the size of the earth. Luckily for future Americans, Christopher Columbus used these estimates to plan his westward voyage from Europe to Asia. If Columbus had known the true magnitude of his quest, he probably would have stayed home.[48]

Rampaging Romans made life difficult for Greek mathematicians and scientists. In 212 BC the brilliant mathematician and scientist, Archimedes (who had once ran nude through the streets shouting 'Eureka' when he had solved a difficult problem), was killed during a Roman invasion of Syracuse. He was immersed in difficult mathematics and did not notice the soldier behind him.[49]

In 47 BC, the Romans destroyed the Library in Alexandria. The Roman General Julius Caesar attacked an Egyptian fleet in the harbor of Alexandria and, in the process, set fire to the

[46] Maor, p. 23.
[47] Boyer, p. 179.
[48] Maor, p. 24-26.
[49] Dunham, p. 85-88.

great library. The collected knowledge of two and a half centuries, the largest book collection in the world, was lost.

The rise of Christianity also challenged mathematics. Greek schools were closed, thousands of books burned, and many mathematicians were murdered. In 529 AD the Roman emperor Justinian ordered all Greek schools of philosophy closed. This included Plato's Academy.[50] St. Augustine summed up the spirit of the times: "The good Christian should beware of mathematicians and all those who make empty prophecies. The danger already exists that the mathematicians have made a covenant with the devil to darken the spirit and to confine man in the bonds of Hell."[51]

As European mathematics decelerated into the Middle Ages, mathematical activity plodded on in India. Indian mathematicians were likely aware of Greek astronomy and trigonometry. The *Siddhāntas*, composed sometime around the year 400, are similar enough to Ptolemy to make this a reasonable assumption. Indian mathematicians made incremental contributions to trigonometric techniques, and set in motion an etymological journey that, through a series of translational mishaps, led to our modern word "sine".[52]

Approaching the Renaissance, Europe began to emerge from the dark. Around 1450, Johann Gutenberg invented the movable type printing press. This accelerated an already

[50] Kline, *Mathematical Thought from Ancient to Modern Times*, p. 180-182.
[51] Kline, *Mathematics in Western Culture*, p. 3.
[52] Boyer, p. 231-237.

changing zeitgeist.[53] Surviving manuscripts of Greek scholarship were translated and published. Europe fell in love with Greek learning.[54]

Johann Müller (1436-1476), better known as Regiomontanus, set up his own printing press and astronomical observatory in Germany.[55] He immersed himself in *The Almagest* and the writings of Hindu and Arab scholars. In 1464 he completed his own treatise on trigonometry, *On Triangles of Every Kind*. Regiomontanus enticed his readers: "You, who wish to study great and wondrous things, who wonder about the movement of the stars, must read these theorems about triangles…For no one can bypass the science of triangles and reach a satisfying knowledge of the stars…A new student should neither be frightened nor despair."[56]

On Triangles of Every Kind was the most important book on trigonometry of its time.[57] It was introduced to the astronomer, Nicholas Copernicus (1473-1543) by his student Rheticus. The two astronomers studied the book together. It was Rheticus who persuaded Copernicus to publish *On the Revolutions of the Heavenly Spheres*[58] in which Copernicus explained his heliocentric (sun centered) universe.[59] This treatise made substantial use of trigonometry and was heavily influenced by the work of Regiomontanus. Rheticus synthesized the ideas of Regiomontanus and Copernicus with ideas of his own in his book *Opus palatinum de triangulis*.

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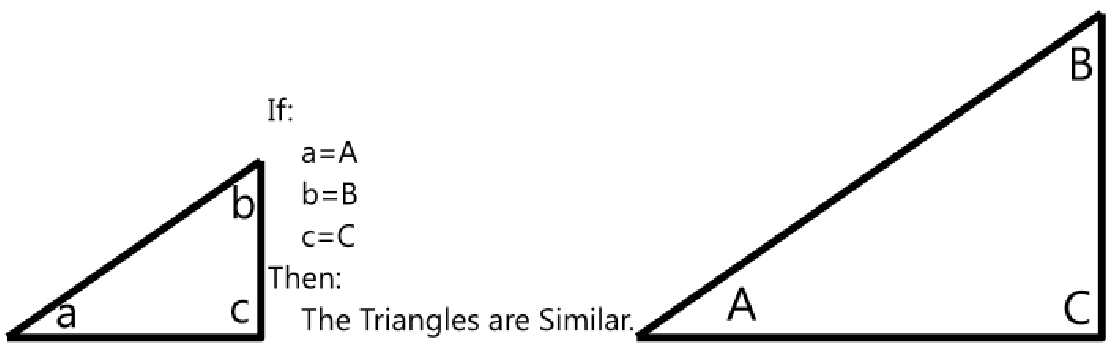
[53] Kline, *Mathematical Thought from Ancient to Modern Times*, p. 217.
[54] Boyer, p. 297-299.
[55] Boyer, p. 301.
[56] Maor, p. 39-45.
[57] Maor, P 40.
[58] Kline, *Mathematics in Western Culture*, p. 100.
[59] Maor, p. 46.

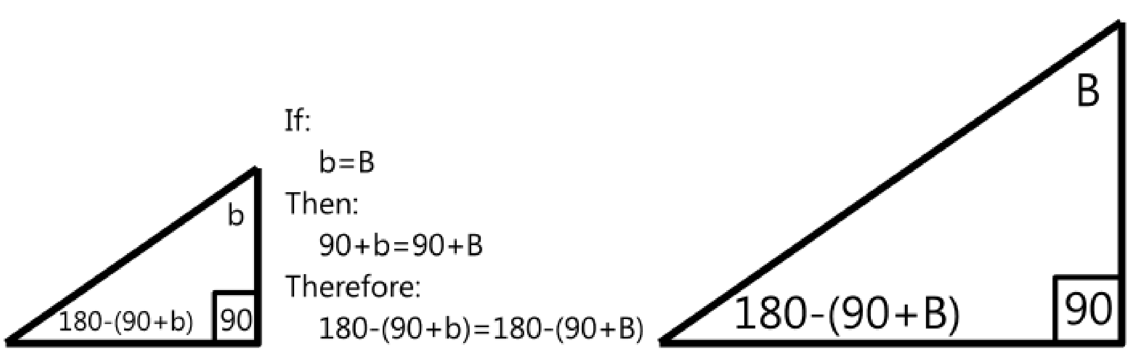
He was the first to discuss all six trigonometric functions, and he evolved trigonometry from the study of arcs of circles into the study of sides of right triangles.[60]

It is with these triangles that we begin the mathematics portion of this paper.

By definition, triangles are similar if their corresponding angles are equal. If it can be proven that two of the angles are equal, it is sufficient to deduce (from the fact that the sum of angles in any triangle equals 180 degrees) that the third angle is also equal and the triangles are similar.[61]



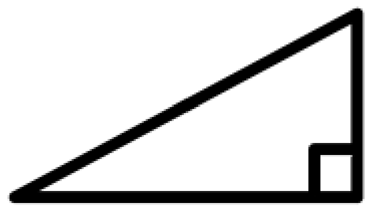
In the case of right triangles (a triangle in which one of the angles is 90 degrees), we need only match up one of the acute angles (an acute angle is an angle of less than 90 degrees) to know that all the angles are equal and the triangles are similar.



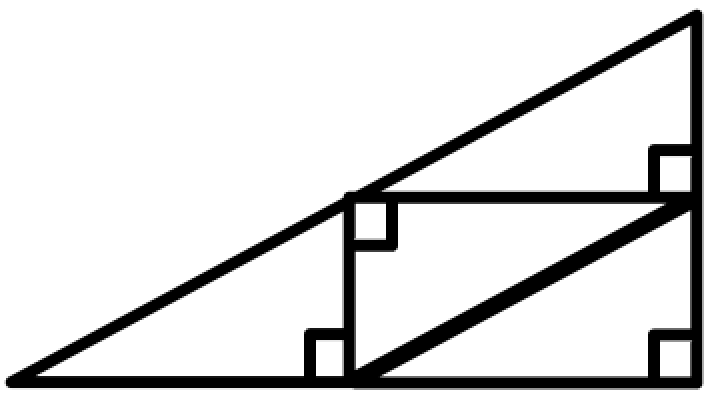
[60] Boyer, p. 320-321.
[61] Dunham, p. 67.

Why do we care if the triangles are similar? If two triangles are similar, then the ratio of the lengths of any two sides of either triangle equals the respective ratio of the other triangle. To help us understand this, let us perform a simple thought experiment.[62]

Consider a right triangle.

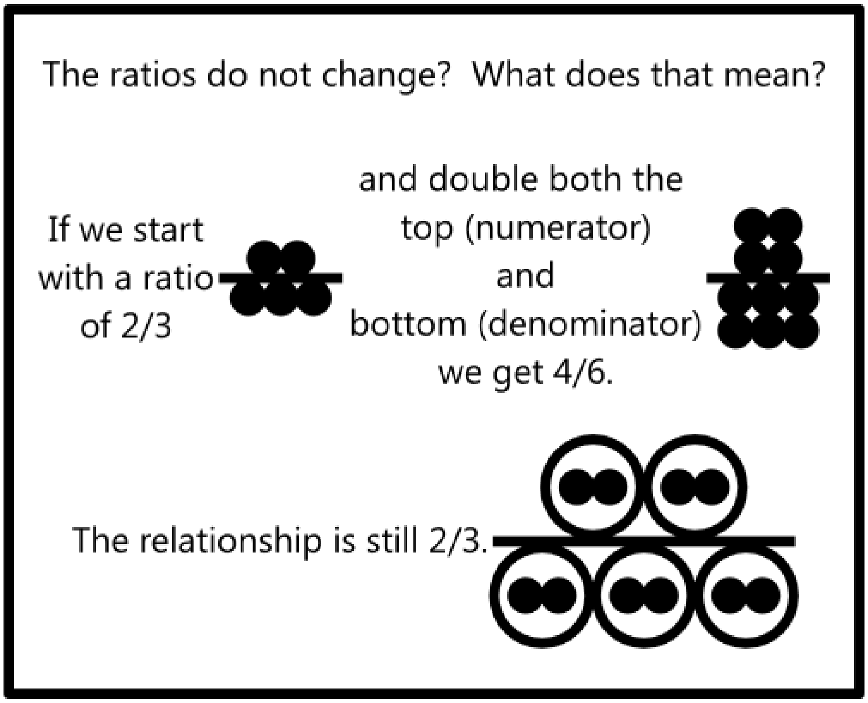


Make one of the sides twice as long while keeping all the angles the same.

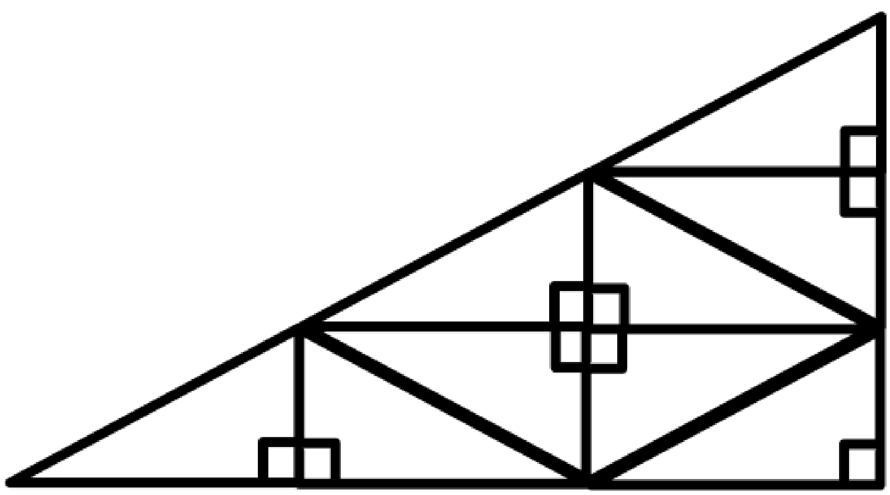


One can see from the illustration above that the only way to double one side without changing any of the angles is to double the other two sides as well. All the sides are now twice as long. Because all the sides have been multiplied by the same amount, the ratios between the sides do not change.

[62] Kline, *Mathematics in Western Culture*, p. 67.

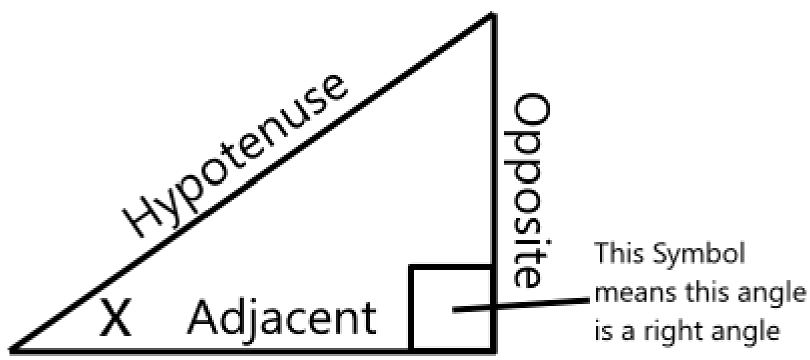


Instead of doubling, let us imagine tripling the original triangle. Remember not to change any of the angles.



Once again, the only way to triple one side without changing any of the angles is to triple the other two sides as well. All the sides are now three times as long. Because all the sides have been multiplied by the same amount, the ratios between the sides remain constant.

What can be done with this information? It turns out, quite a lot. Consider the illustration below. We are now in a position to make this statement: For any size right triangle with angle X, the ratio of the lengths of the side labeled “Opposite” to the side labeled “Hypotenuse” remains constant.



What do we mean by the hypotenuse? Wolfram|Alpha defines hypotenuse as: “The hypotenuse of a right triangle is the triangle’s longest side, i.e., the side opposite the right angle.”[63]

This ratio of the opposite over the hypotenuse is called the Sine.

$$\text{Sine X} = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

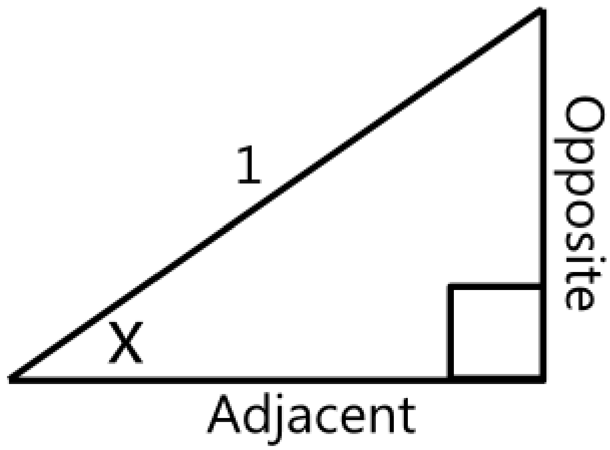
Other ratios are given other names:

$$\text{Cosine X} = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

$$\text{Tangent X} = \frac{\text{Opposite}}{\text{Adjacent}}$$

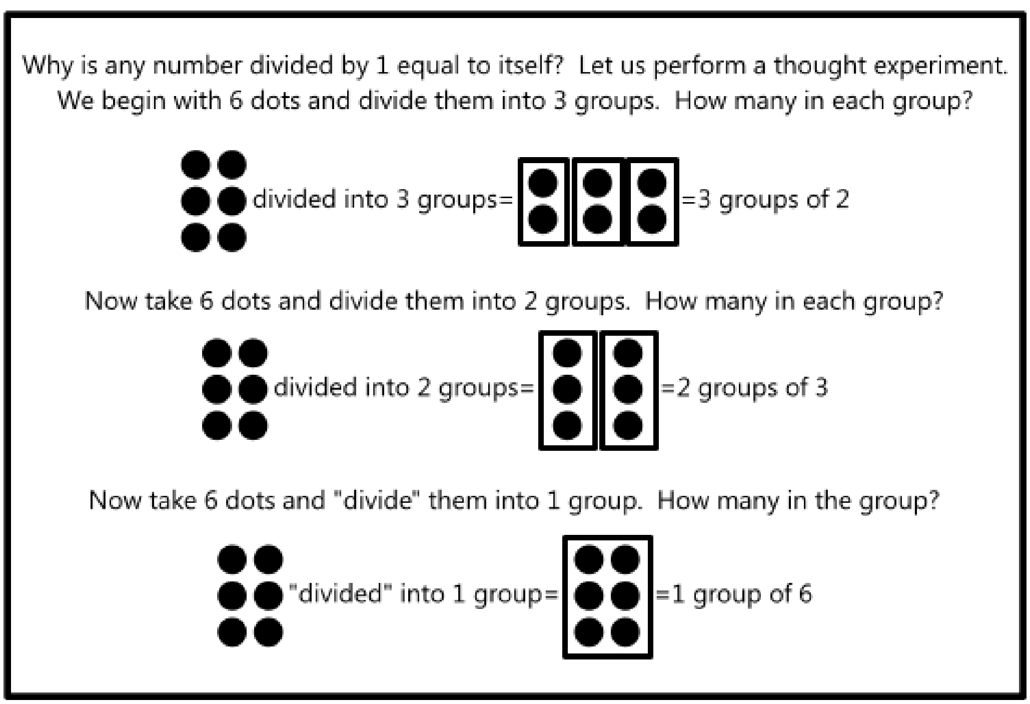
[63] Wolfram|Alpha (April 28, 2011) http://www.wolframalpha.com/input/?i=hypotenuse

These simple ratios are the basic building blocks of Trigonometry. Let us build a triangle with a Hypotenuse 1 unit in length.

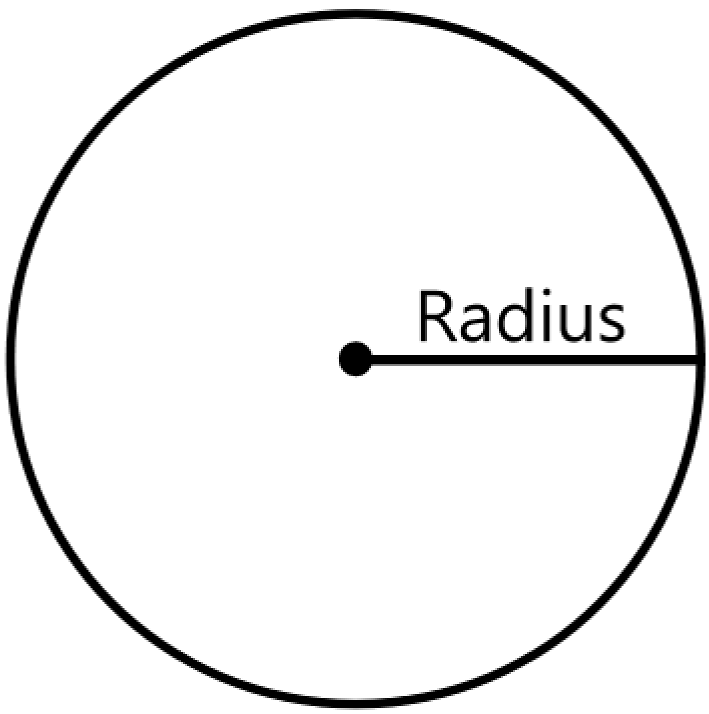


We recall from basic arithmetic that anything divided by 1 equals itself. Therefore, when dealing with the specific case of a right triangle with a hypotenuse of one unit:

$$\text{Sine X}=\frac{\text{Opposite}}{1} \quad \text{Therefore:} \quad \text{Sine X}=\text{Opposite}$$



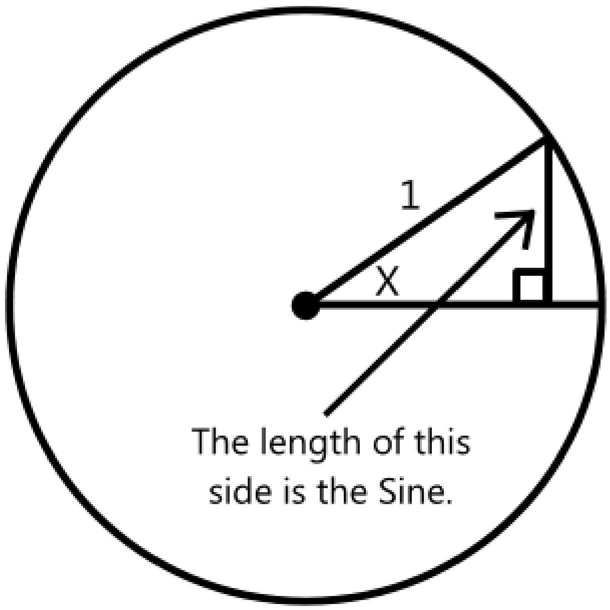
We will set aside triangles for one moment and introduce our next ingredient: the circle. The Oxford English Dictionary defines a circle: "A plane figure bounded by a single curved line, called the circumference, which is everywhere equally distant from a point within, called the centre."[64]



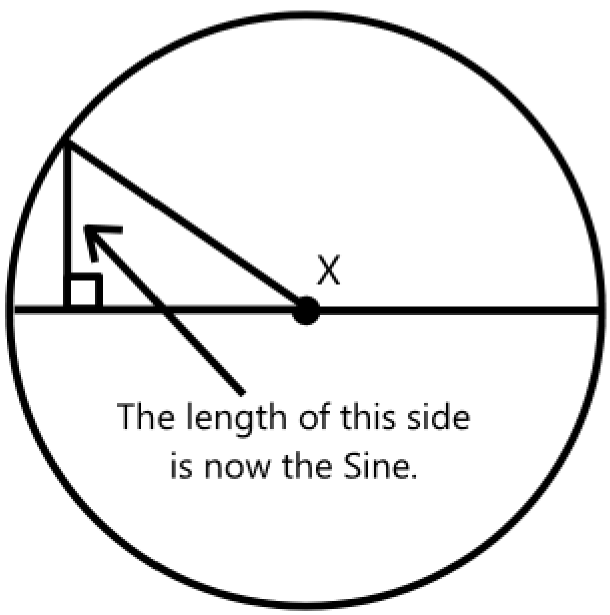
The important concept to understand is that every point on the circumference (the line that makes up the outside of the circle) is equidistant (the same distance) from the center point. A unit circle is a circle with a radius (the distance from the center to the circumference) of one unit.

[64] "circle, n." in *Oxford English Dictionary Online* (November 2010) <http://www.oed.com:80/Entry/33187> (accessed February 23, 2011).

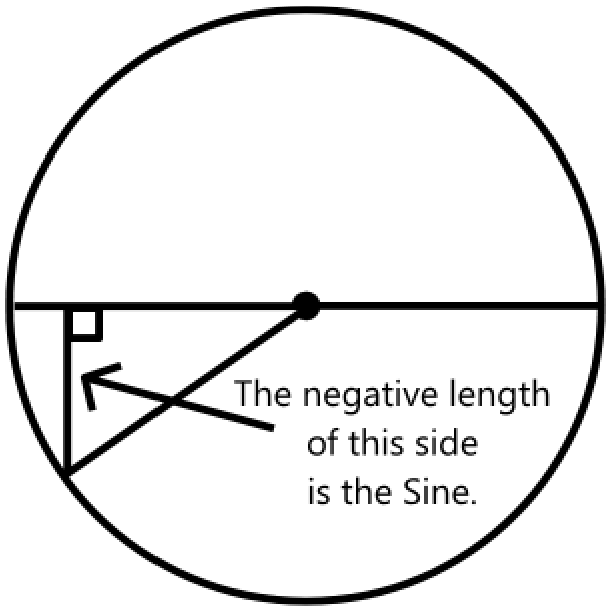
Let us now combine the unit circle with the right triangle. Let us use a right triangle with a hypotenuse of one unit in length. This hypotenuse is also the radius of the circle.



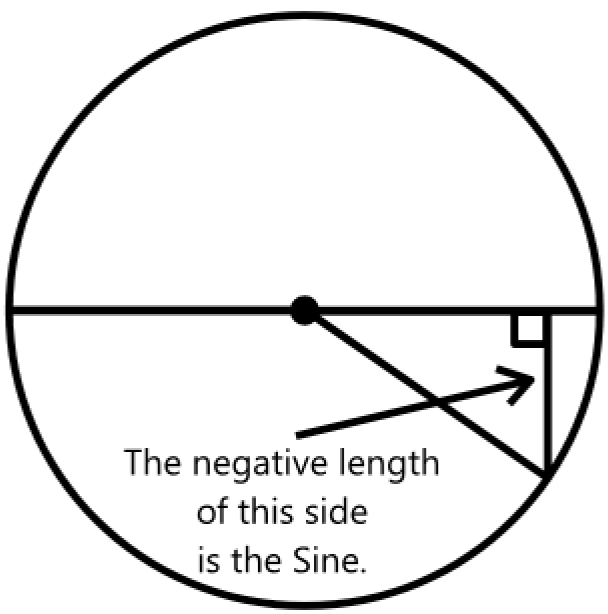
As angle X varies between 0 and 90 degrees, the Sine of X will vary between 0 and 1. To find the Sine of an angle between 90 and 180 degrees, flip the triangle around.



To find the Sine of an angle between 180 and 270 degrees, flip the triangle again. This length should be represented as a negative number between 0 and -1.



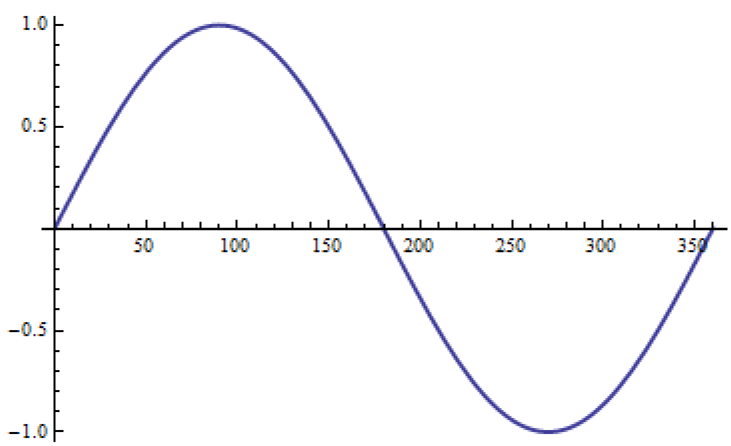
To find the Sine of an angle between 270 and 360 degrees, flip the triangle again. This length also should also be represented as a negative number between 0 and -1.



One more flip and the triangle is back to its original position.[65]

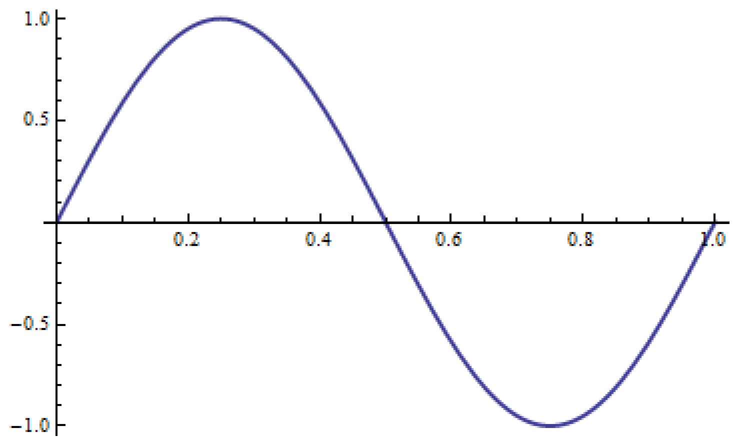
As angle X varies between 0 and 360 degrees, the Sine will vary between 1 and -1. This relationship between the angle X and the value of the Sine can be expressed in mathematical symbolism as **sin x** although many computer programming languages require **Sin(x)**.

Let us create a graph with **x** on the horizontal axis and **Sin(x)** on the vertical. We will do this through one complete revolution: 360 degrees.

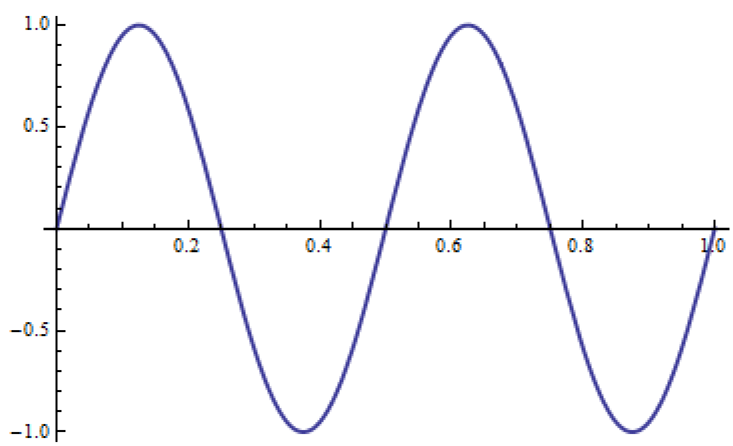


[65] Transnational College of LEX, *Who is Fourier?* (Belmont, MA: Language Research Foundation, 1995), p. 23-26.

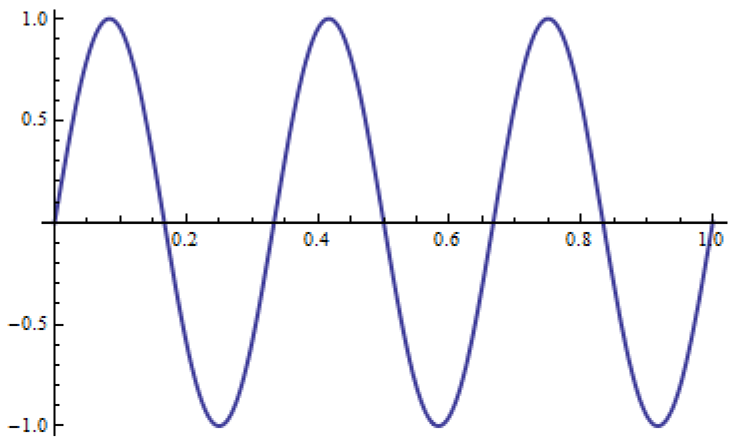
Because we are relating this to sound, we should convert our equation to represent frequency, or vibrations per second. If we let **t** represent time in seconds, then **Sin (360t)** will give us one revolution per second, or one hertz. By multiplying **t** by 360, each second equals 360 degrees of rotation, one complete circle. Below is a graph with seconds on the horizontal axis and **Sin (360t)** on the vertical.



To get 2 hertz, we need 2 revolutions per second. Therefore we must multiply 360 by 2. We get **Sin (720t)**. The graph is below.



To get 3 hertz, we need 3 revolutions per second. Therefore we multiply 360 by 3. We get **Sin (1080t)**. The graph is below.



(To be continued…)

Bibliography

Adkins, Cecil. "Monochord." *Grove Music Online, Oxford Music Online.* http://www.oxfordmusiconline.com/subscriber/article/grove/music/18973 (accessed March 21, 2011).

Borwick, John. "Hertz." *Oxford Music Online.* http://www.oxfordmusiconline.com/subscriber/article/opr/t114/e3231 (accessed March 30, 2011).

Boyer, Carl B. *History of Mathematics.* New York: John Wiley & Sons, Inc, 1968.

Bruhn, Siglind. *The Musical Order of the World.* Hillsdale, NY: Pendragon Press, 2005.

"circle, n." *Oxford English Dictionary Online.* November 2010. <http://www.oed.com:80/Entry/33187> (accessed February 23, 2011).

Dunham, William. *Journey through Genius: The Great Theorums of Mathematics.* New York, New York: Penguin Group, 1990.

Feldman, Marian H. "Mesopotamia." *The Oxford Encylopedia of Ancient Egypt.* 2005. ://www.oxford-ancientegypt.com/entry?entry=t176.e0453 (accessed April 7, 2011).

Gunther Holbl, Elizabeth Schwaiger. "The Oxford Encylcopedia of Ancient Egypt." *Ptolemaic Period.* 2005. www.oxford-ancientegypt.com/entry?entry=t176.e0586 (accessed February 28, 2011).

Helmholtz, Hermann L. F. *On The Sensation of Tone as a Physiological Basis for the Theory of Music.* New York: Dover Publications, INC., 1954.

Isaacson, Walter. *Einstein His Life and Universe.* New York, NY: SIMON & SCHUSTER, 2007.

Jeans, James. *Science & Music.* New York: Dover Publications, 1968.

Johnston, Ian. *Measured Tones: The interplay of physics and music.* New York, NY: IOP Publishing Ltd., 1989.

Kline, Morris. *Mathematical Thought from Ancient to Modern Times.* New York: Oxford University Press, 1972.

—. *Mathematics in Western Culture.* New York: Oxford University Press, 1953.

Maor, Eli. *Trigonometric Delights.* Princeton, New Jersey: Princeton University Press, 1998.

Transnational College of LEX. *Who is Fourier? A Mathematical Adventure.* Belmont,MA: Language Research Foundation, 1995.

Wolfram|Alpha. *Wolfram|Alpha.* February 27, 2011. http://www.wolframalpha.com/input/?i=chord (accessed February 27, 2011).

—. *Wolfram|Alpha.* April 26, 2011. http://www.wolframalpha.com/input/?i=newton+third+law+of+motion (accessed April 26, 2011).

—. *Wolfram|Alpha.* April 26, 2011. http://www.wolframalpha.com/input/?i=conservation+of+energy (accessed April 26, 2011).

—. *Wolfram|Alpha.* April 26, 2011. http://www.wolframalpha.com/input/?i=kinetic+energy+definition (accessed April 26, 2011).

—. *Wolfram|Alpha.* April 26, 2011. ://www.wolframalpha.com/input/?i=elastic+potential+energy (accessed April 26, 2011).

—. *Wolfram|Alpha.* April 28, 2011. http://www.wolframalpha.com/input/?i=hypotenuse (accessed April 28, 2011).